

M.Sc.(Maths.) Semester - 4 (CBCS) Examination
March/April - 2025 (NEW COURSE)
Integration Theory (Core)

Time: 2:30 Hours

Marks: 70

Instructions:

1. All questions are compulsory.
2. Figures to the right indicate marks.

Que1(A) Answer the following questions. (04)

- (1) Let μ be a measure on a measurable space $(X, \mathcal{A})$. If E and F are measurable subsets of X , $E \subset F$ and if $\mu(F) < \infty$, then show that $\mu(F - E) = \mu(F) - \mu(E)$.
- (2) Give an example of measure space $(X, \mathcal{A}, \mu)$ which is not complete and give appropriate reason.

Que-1(B) Answer any two questions out of three. (10)

- (1) Let $\{f_n\}$ be a sequence of measurable functions on a measure space $(X, \mathcal{A}, \mu)$ converging to a function f (pointwise) on X . Let $E \in \mathcal{A}$. If g is integrable over E and $|f_n| \leq g$ on E for all n , then show that $\int_E f d\mu = \lim_n \int_E f_n d\mu$.
- (2) Let $\{f_n\}$ be a sequence of nonnegative measurable functions on a measure space $(X, \mathcal{A}, \mu)$, then show that $\int_X \left(\liminf_n f_n \right) d\mu \leq \liminf_n \int_X f_n d\mu$.
- (3) Show that every σ - finite measure space is saturated.

Que-2(A) Answer the following questions. (04)

- (1) Let v_1 and v_2 be finite signed measure on a measurable space $(X, \mathcal{A})$, then show that $|v_1 + v_2| \leq |v_1| + |v_2|$.
- (2) Let v be a signed measure on a measurable space $(X, \mathcal{A})$, and let A and B be positive sets. Then show that $A \cup B$ is a positive set.

Que-2(B) Answer any two questions out of three. (10)

- (1) Let v be a signed measure on a measurable space $(X, \mathcal{A})$, then show that there exist a positive set A and negative set B such that $X = A \cup B$ and $A \cap B = \emptyset$.
- (2) Let v be a signed measure on a measurable space $(X, \mathcal{A})$, then show that the Jordan decomposition of v is unique.
- (3) Let v be a signed measure on a measurable space $(X, \mathcal{A})$, and let E be a measurable subset of X such that $0 < v(E) < \infty$. Show that E contains a positive set F such that $v(F) > 0$.

Que-3(A) Answer the following questions. (04)

- (1) Let v_1 and v_2 be finite signed measures on a measurable space $(X, \mathcal{A})$, and μ be a measure on $(X, \mathcal{A})$. If $v_1 \ll \mu$ and $v_2 \ll \mu$, then show that $v_1 + v_2 \ll \mu$.
- (2) Let v be a signed measure and μ be a measure on a measurable space $(X, \mathcal{A})$. If $v \perp \mu$ and $v \ll \mu$, then show that $v = 0$.

Que-3(B) Answer any two questions out of three. (10)

- (1) Let v and μ be a σ - finite measure on a measurable space $(X, \mathcal{A})$, then show that there exist a pair of measures v_0 and v_1 such that $v_0 \perp \mu$, $v_1 \ll \mu$ and $v = v_0 + v_1$.
- (2) Let v_1 and v_2 be measures and μ be a σ - finite measure on a measurable space $(X, \mathcal{A})$, and let $v_1 \ll \mu$ and $v_2 \ll \mu$. Then show that $\left[\frac{d(v_1+v_2)}{d\mu} \right] = \left[\frac{dv_1}{d\mu} \right] + \left[\frac{dv_2}{d\mu} \right]$.
- (3) Let v be a signed measure and μ be a σ - finite measure on a measurable space $(X, \mathcal{A})$, and let $v \ll \mu$. If f is nonnegative measurable function on X , then show that $\int_E f dv = \int_E f \left[\frac{dv}{d\mu} \right] d\mu$ for every $E \in \mathcal{A}$.

Que-4(A) Answer the following questions. (04)

- (1) Define Baire measure on a real line.
- (2) Let $(X, \mathcal{A}, \mu)$ be a measure space, and a measurable function f is bounded a.e. $[\mu]$ on X . Then show that f is finite a.e. $[\mu]$ on X .

Que-4(B) Answer any two questions out of three. (10)

- (1) Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $1 \leq p < r \leq \infty$. Then show that $L^r(\mu) \subset L^p(\mu)$.
- (2) Let $1 \leq p < \infty$. Let $f \in L^p(\mu)$, and let $\epsilon > 0$. Then show that there is a measurable simple function s vanishing outside a set of finite measure such that $\|f - s\|_p < \epsilon$.
- (3) Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $1 \leq p < \infty$. If $f, g \in L^p(\mu)$, then show that $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

Que-5(A) Answer the following questions. (04)

- (1) Let X be a locally compact Hausdorff space, and let M be a σ -algebra on X containing all Baire sets. When is $E \in M$ called outer regular for μ ?
- (2) State Riesz-Markov theorem.

Que-5(B) Answer any two questions out of three. (10)

- (1) Let μ be a Baire measure on a locally compact space X , and let E be a σ -bounded Baire set in X . If $\epsilon > 0$, then show that there is a σ -compact open set O containing E such that $\mu(O - E) < \epsilon$.
- (2) Let μ be a finite measure defined on a σ -algebra M which contains all the Baire sets of a locally compact space X . If μ is inner regular, then show that it is regular.
- (3) Let μ^* be a topologically regular outer measure on a locally compact Hausdorff space X . Show that each Borel set is μ^* -measurable.
